UBC MATH CIRCLE 2024 PROBLEM SET 1 SOLUTIONS

Problem 1. *Find all pairs of integers* (a, n) *for which the following holds.*

$$\frac{(a+1)^n - a^n}{n} \in \mathbb{Z}. \tag{1}$$

Solution. The only pairs of integers for which (1) holds are $(a, n) = (a, 1)$ for $a \in \mathbb{Z}$. Indeed, suppose that $n > 1$. Let p be the smallest prime such that $p \mid n$. Evidently if $p \mid a$ then $(a+1)^n - a^n \equiv 1 \mod p$ and (1) fails. Hence we may assume that $a \not\equiv 0 \mod p$. Let $b = a^{-1} \mod p$. We have that

$$(a+1)^n \equiv a^n \mod p$$

and multiplying by b^n gives

$$((a+1)b)^n \equiv 1 \mod p.$$

In particular, $(a+1)b \not\equiv 0 \mod p$. Let d be the order of $(a+1)b \mod p$ (i.e. the smallest integer such that $((a+1)b)^d \equiv 1 \mod p$). The computation above shows that $d \mid n$, but we also know that

$$((a+1)b)^{p-1} \equiv 1 \mod p$$

by Fermat's little theorem. Therefore $d \mid (p-1)$ and $d < p$. Since p is the smallest prime factor of n and $d \mid n$, it follows that $d = 1$. But then $(a+1)b \equiv 1 \mod p$ and $a + 1 \equiv a \mod p$ which is a contradiction. Thus $n = 1$.

Problem 2. *Find all integers* x *and* y *for which* $x^3 - y^2 = 9$. *(As a bonus problem, what happens when* 9 *is replaced by* 7*?)*

Solution. We show that $x^3 - y^2 = 9$ has no integer solutions. First note that x, y have opposite parity since otherwise the left hand side is even. We split the proof into three cases: (i) x is even and y is odd, (ii) $x \equiv -1 \mod 4$ and y is even, and (iii) $x \equiv 1 \mod 4$ and y is even.

(i) (x is even and y is odd) Then $-y^2 \equiv x^3 - y^2 \equiv 1 \mod 4$. Since -1 is not a square modulo 4, there are no solutions.

(ii) ($x \equiv -1 \mod 4$ and y is even) Then $(-1)^3 \equiv x^3 - y^2 \equiv 1 \mod 4$ which is clearly false, hence this case also has no solutions.

(iii) ($x \equiv 1 \mod 4$ and y is even) We rewrite the equality $x^3 - y^2 = 9$ as

$$y^2 + 1 = x^3 - 8 = (x-2)(x^2 + 2x + 4).$$

For any $y \in \mathbb{Z}$, the left hand side is positive, hence $x \geq 3$. Since $x \equiv 1 \mod 4$, we have $x \geq 5$ and therefore $x - 2 \geq 3$. Thus there exists a prime $p \mid (x-2)$. Reducing the equality above modulo p gives

$$y^2 \equiv -1 \mod p$$

Since $x - 2$ is odd, p is odd. Thus p is an odd prime for which -1 is a square modulo p and we conclude that $p \equiv 1 \mod 4$. However, the choice of $p \mid (x-2)$ was entirely arbitrary. Thus if $x - 2 = \prod_{i=1}^{r} p_i$ is the prime factorization of $x - 2$, then $p_i \equiv 1$

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mod 4 for all $i \in \{1, \dots, r\}$. In particular, $x - 2 \equiv 1 \mod 4$ and $x \equiv 3 \mod 4$ which is a contradiction.

Problem 3. *Find all polynomials $f \in \mathbb{R}[x]$ such that for all real numbers a, b, c satisfying $ab + bc + ca = 0$, we have*

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c).$$

Solution. Setting $a = b = c = 0$ implies that $3f(0) = 2f(0)$ and $f(0) = 0$. Now taking $b = c = 0$ and letting a vary shows that $f(a) + f(-a) = 2f(a)$ so that $f(a) = f(-a)$ for all $a \in \mathbb{R}$. In particular, every monomial of f has even degree. We now make the substitution $x = a - b$, $y = b - c$, $z = c - a$ and $w = a + b + c$ (so that $x + y + z = 0$). The given condition becomes $2w^2 = x^2 + y^2 + z^2$. Therefore, the original functional equation is equivalent to finding all polynomials $f \in \mathbb{R}[x]$ with $f(0) = 0$ and satisfying

$$f(x) + f(y) + f(x+y) = 2f(\sqrt{x^2 + xy + y^2}) \text{ for all } x, y \in \mathbb{R}.$$

Here we have used the fact that $f(z) = f(-x-y) = f(x+y)$ as well as the equality $f(w) = f(\sqrt{(x^2+y^2+z^2)/2}) = f(\sqrt{x^2+xy+y^2})$. In particular, the left and right hand side define the same polynomial. Suppose that $\deg f = 2n$. Equating monomials of degree $2n$, we find that

$$x^{2n} + y^{2n} + (x+y)^{2n} = 2(x^2 + xy + y^2)^n.$$

Taking $y = tx$ and factoring out x^{2n} we should have

$$1 = t^{2n} + (1+t)^{2n} = 2(1 + t + t^2)^{2n}$$

for all $t \in \mathbb{R}$. Assume $n \geq 3$. Let $\omega = \frac{-1+\sqrt{-3}}{2}$ be a primitive 3-rd root of unity. Since $1 + \omega + \omega^2 = 0$ and $n \geq 3$, it follows that ω is a root of the right hand side with multiplicity at least 3. Hence it must be a root of the left hand side with multiplicity at least 3. Let g denote the polynomial on the left hand side. Then

$$\begin{aligned} g''(\omega) &= 2n(2n-1)(\omega^{2n-2} + (1+\omega)^{2n-2}) \\ &= 2n(2n-1)(\omega^{2n-2} + (-\omega^2)^{2n-2}) \\ &= 2n(2n-1)\omega^{2n-2}(1 + \omega^{2n-2}) \\ &\neq 0 \end{aligned}$$

In particular ω is not a triple root of g, a contradiction. Hence $\deg f \leq 4$ so that $f = ax^4 + bx^2$ are the only possible solutions. Clearly any linear combination of solutions is also solution, hence it is enough to check that x^4 and x^2 are solutions. Indeed

$$\begin{aligned} x^2 + y^2 + (x+y)^2 &= 2(x^2 + xy + y^2) \\ x^4 + y^4 + (x+y)^4 &= 2(x^2 + xy + y^2)^2 \end{aligned}$$

which concludes the proof.