UBC MATH CIRCLE 2024 PROBLEM SET 2 SOLUTIONS

Problem 1. *Let $f \in \mathbb{Z}[x]$ be a polynomial with integer coefficients and let $S \subset \mathbb{Z}$ be a finite set of positive integers such that for any $n \in \mathbb{Z}$, there is a $s \in S$ such that $s \mid f(n)$. Show that there is an $s \in S$ such that $s \mid f(n)$ for all $n \in \mathbb{N}$.*

Proof. Suppose for contradiction that S does not satisfy the required condition. Clearly $1 \notin S$. For any $s \in \mathbb{S}$, there is an $n \in \mathbb{Z}$ such that $s \nmid f(n)$. Thus there is a prime divisor p of s such that $p \nmid f(n)$. Moreover, if $m \in \mathbb{Z}$ with $s \mid f(m)$, then also $p \mid f(m)$. In particular, replacing s with p does not change the set of $m \in \mathbb{Z}$ such that $s \mid f(m)$. Hence we can, in this way, reduce S to a set of primes T. Then for any $p \in T$, there is a $k_p \in \mathbb{Z}$ such that $p \nmid f(k_p)$. Whenever $n \equiv k_p \mod p$, we have $p \nmid f(n)$. By the Chinese remainder theorem, there is an $m \in \mathbb{Z}$ such that $m \equiv k_p \mod p$ for all $p \in T$. In particular, $p \nmid f(m)$ for all $p \in T$, which is a contradiction. □

Problem 2. *Is it possible to cover the plane with the interiors of a finite number of parabolas?*

Proof. This is impossible. Indeed, if $\mathcal{P}$ is a parabola and $P \subset \mathbb{R}^2$ is its interior, then for any line $\ell \subset \mathbb{R}^2$, the intersection $\ell \cap P$ has infinite length if and only if ℓ is parallel to the axis of symmetry of $\mathcal{P}$. Hence we let $L = \{\ell_n : n \in \mathbb{N}\}$ be a collection of lines in $\mathbb{R}^2$ such that ℓ_n and ℓ_m are not parallel for all $n \neq m$ (we could for example take ℓ_n to have slope n). Then for any finite collection $\mathscr{P} = \{\mathcal{P}_i\}_{1 \leq i \leq k}$, there is an $N \in \mathbb{N}$ such that the line ℓ_N is not parallel to the axes of symmetry of $\mathcal{P}_i$ for any $i \in \{1, \ldots, k\}$. In particular, the intersection $\ell \cap P_i$ has finite length for all i and $\ell \not\subset \bigcup_{i=1}^k P_i$. □

Problem 3. *Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called* neighbors *if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?*

(1) *If p is in S, then none of the neighbors of p is in S.*
(2) *If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p in S.*

Proof. We show that such a set exists for every n. Define a function $f : \mathbb{Z}^n \to \mathbb{Z}/(2n+1)\mathbb{Z}$ (the residue classes modulo $2n+1$) by

$$f(x_1, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n \mod 2n+1$$

and let $S = f^{-1}(0)$. To check condition (1), we note that if $p \in S$ and q is a neighbor of p differing only in the i-th coordinate then

$$f(q) = f(p) \pm i \equiv \pm i \mod 2n+1$$

and so $q \notin S$. Conversely, if $p \in \mathbb{Z}^n$ with $p \notin S$ then there is a unique $i \in \{1, \ldots, n\}$ such that either $f(p) \equiv i \mod 2n+1$ or $f(p) \equiv -i \mod 2n+1$ (but not both). In the first case, we let q be the element created by subtracting 1 from the i-coordinate of p and in the second case by adding 1 to the i-coordinate of p. Then $q \in S$ and the construction above shows that q is unique. □

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