UBC MATH CIRCLE 2024 PROBLEM SET 3 SOLUTIONS

Problem 1. *On an infinite square grid we place finitely many trains, which each occupy a single cell and face in one of the four cardinal directions. Trains may never occupy the same cell. It is given that the cell immediately in front of each train is empty, and moreover no two trains face towards each other (no right-facing train is to the left of a left-facing train within a row, etc.). In a move, one chooses a train and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each train infinitely many times.*

Remark. The idea for the solution is quite simple: we clear the even rows of vertical trains and move the horizontal trains on those rows far out, then clear the odd rows of vertical trains and move the remaining horizontal trains far out. Finally we move the vertical trains far out. We have now essentially separated any interactions between vertical and horizontal trains. So we can now move any of the trains without having to worry about interactions. Actually formalizing this though is quite challenging as the following long solution demonstrates.

Solution. We represent points on the infinite square grid by vectors of integers $\vec{m} = (m, n) \in \mathbb{Z}^2$ and directions by vectors $\vec{v} = \{\pm(0, 1), \pm(1, 0)\}$. Let $\{(\vec{m}_i, \vec{v}_i)\}_{i=1}^n$ denote the positions and directions of the n trains so that the i train lies at $\vec{m}_i = (m_i, n_i) \in \mathbb{Z}^2$ and has direction $\vec{v}_i$ (in particular, moving the i-train k-steps in its given direction can be achieved by replacing its position $\vec{m}_i$ with $\vec{m}_i + \vec{v}_i$). Let $N \in \mathbb{N}$ be chosen such that $N > \max\{|m_i|, |n_i|\} + 1$ for all $1 \leq i \leq n$. We construct the sequence as follows. First, proceed down the list of trains $\{1, \ldots, n\}$ moving i-th train if its direction $\vec{v}_i = \pm(0, 1)$ (i.e. it points either North or South) and $2 \mid n_i$. Since the cell immediately in front of a train is empty and since moving a train vertically cannot affect any other trains moving vertically, all of these moves are possible. In other words, we may assume that in the starting configuration, there are no trains $i \in \{1, \ldots, n\}$ such that $\vec{v}_i = \pm(0, 1)$ and with $2 \mid n_i$. In particular, any horizontal train i having $2 \mid n_i$ cannot be obstructed by a vertical train. We proceed down the even rows (those having $2 \mid n_i$) in each one moving the horizontal trains until $|m_i| > N$ (we can do this one move at a time since the square in front of a train is never obstructed). Since no trains point towards each other, these movements are always possible. Hence we may assume that no vertical train is obstructed by a horizontal train i having $2 \mid n_i$. We can now move the vertical trains on odd rows $2 \mid n_i - 1$ one square forward and repeat the argument above with the horizontal trains in odd rows. Thus we may assume that in fact no horizontal train is obstructed by a vertical train and vice versa. hence we may move each of the vertical trains one square forward at a time until $|n_i| > N$ for all of them. Finally, moving any vertical or horizontal train one square forward in their corresponding direction places them one square further away from the original $N \times N$ box which held them at the beginning of the problem. Hence moving each of the trains forward again results in a position where no horizontal train is obstructed by a vertical train and vice versa. Hence we may no take the sequence to be $\{1, \ldots, n, 1, \ldots, n, 1, \ldots\}$ repeated ad-nauseam. As we noted, this never results in obstructions between horizontal and vertical trains and also can never result

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in obstructions between vertical and vertical or between horizontal and horizontal trains because we move each train forward exactly once before moving any a second time.

Problem 2. *Let m and n be positive integers. Suppose that a given rectangle can be tiled by a combination of horizontal $1 \times m$ strips and vertical $n \times 1$ strips. Prove that it can be tiled using only one of the two types.*

Solution. Assume that the dimensions of the rectangle are $a \times b$. It is clear that both a and b are positive integers. We want to show that either a is divisible by m or b is divisible by n. Let $\zeta = e^{2\pi i/m}$ and $\xi = e^{2\pi i/n}$ be the m-th and n-th roots of unity respectively (these are solutions to $x^m = 1$ respectively $x^n = 1$). Divide the rectangle into ab unit squares and write the number $\zeta^x \xi^y$ into the square in the x-th column and y-th row. For each vertical strip of dimensions $n \times 1$ in the rectangle $a \times b$, the sum of the numbers written into it is

$$\zeta^x \xi^y (1 + \xi + \xi^2 + \ldots + \xi^{n-1}) = \zeta^x \xi^y \cdot \frac{\xi^n - 1}{\xi - 1} = 0$$

Where (x, y) denotes the position of the top of the vertical strip. Analogously, the sum of the numbers in any horizontal strip is

$$\zeta^x \xi^y (1 + \zeta + \zeta^2 + \ldots + \zeta^{m-1}) = \zeta^x \xi^y \cdot \frac{\zeta^m - 1}{\zeta - 1} = 0$$

Since the rectangle is tiled by these strips, the sum of all of the numbers in the rectangle is 0. But the sum is also

$$\sum_{x=1}^{a} \sum_{y=1}^{b} \zeta^x \xi^y = (\zeta + \zeta^2 + \ldots + \zeta^a)(\xi + \xi^2 + \ldots \xi^b) = \zeta\xi \cdot \frac{\zeta^a - 1}{\zeta - 1} \cdot \frac{\xi^b - 1}{\xi - 1}$$

Hence we must have $\zeta^a = 1$ or $\xi^b = 1$ implying that either $m \mid a$ or $n \mid b$. Thus the rectangle may be covered by only one of the two strips.

Problem 3. *A total of 119 residents live in a building with 120 apartments. We call an apartment* overpopulated *if there are at least 15 people living there. Every day, the inhabitants of an overpopulated apartment have a quarrel and each goes off to a different apartment in the building. Is it true that this process will eventually terminate? (Bonus: What about if there are 120 residents?)*

Solution. Let $p_1, \ldots, p_{120}$ denote the 120 apartments, and let a_i denote the number of residents in apartment p_i. Suppose that at the beginning of each day (before any quarrels) and after each quarrel, all of the residents in an apartment shake hands with each other (in particular we are assuming that the quarrels happen one after another and not at the same time). Let S denote the number of handshakes. With the notation above

$$S = \binom{a_1}{2} + \binom{a_2}{2} + \ldots + \binom{a_{120}}{2}$$

If all $a_i < 15$ then the process is completed and we are done. Hence we may assume without loss of generality that $a_1 \geq 15$ and that p_1 is the first apartment to quarrel. suppose that the people in apartment p_1 go off to apartments $p_{i_1}, \ldots, p_{i_{a_1}}$. After the quarrel, the number of handshakes is changed by an amount equal to

$$a_{i_1} + a_{i_2} + \ldots + a_{i_{a_1}} - \binom{a_1}{2}$$

Since $a_{i_1} + a_{i_2} + \ldots + a_{i_{a_1}} \leq 119 - a_1 \leq 119 - 15 = 104$ while $\binom{a_1}{2} \geq \binom{15}{2} = 105$, it follows that S strictly decreases after each quarrel. But S is nonnegative, hence there can only be finitely many quarrels. In other words, the process is eventually completed.

If there are 120 residents, we arrange them into apartments $p_1, \ldots, p_{15}$ placing n people into apartment p_n for $1 \leq n \leq 15$. Apartment p_{15} quarrels and sends 1 person to each of the apartments $p_1, \ldots, p_{14}$ as well as one person to apartment p_{16}. Relabelling, we find ourselves back in the same position as before. Repeating the process above, we find that with 120 residents, the process is not guaranteed to be completed.