UBC MATH CIRCLE 2024 PROBLEM SET 4

Problem 1. *For a positive integer n, define $S_n = 1! + 2! + \ldots + n!$. Prove that there exists an integer n such that S_n has a prime divisor greater than 10^{2024}.*

Problem 2. *Prove or Disprove: The only polynomials $f \in \mathbb{Q}[x]$ which induce a bijection $\mathbb{Q} \to \mathbb{Q}$ are linear.*

Problem 3. *A subset $S \subset \mathbb{Z}$ is called* square *if $a + b$ is a perfect square for any distinct $a, b \in S$. Two square subsets $S, T \subset \mathbb{Z}$ are called* equivalent *if there is a rational number $\alpha \in \mathbb{Q}$ such that $S = \alpha^2 T = \{\alpha^2 t : t \in T\}$. Prove that there are infinitely many inequivalent square subsets $S \subset \mathbb{Z}$ with $\#S = 4$.*

Remark. One can generalize this problem in a few ways, all of which very quickly lead to areas of active research. For example replacing squares by cubes still gives infinitely many inequivalent solutions, of which the first is $(-217792516, 255052220, 312611332, 350443516)$, degree 4 is unknown and degree 5 would be disproved by a generalization of Fermat's Last Theorem. One could also restrict to squares but ask that $\#S > 4$. In this case, it is known that there are infinitely many inequivalent square sets with $\#S = 5$, of which the first is $(7442, 28658, 148583, 177458, 763442)$. For $\#S = 6$ only one solution is known:

$$(339323777731946898, 1393697157060854002, 2146648434867118098,$$
$$8397374854916636127, 129829308441197954098, -303704776155745998)$$

Beyond this is unknown. In all of these cases, a very big conjecture in arithmetic geometry (the Bombieri Lang conjecture) implies that the size of such sets is bounded.

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