UBC MATH CIRCLE 2024 PROBLEM SET 5 SOLUTIONS

Problem 1. *Basketball star Shanille OKeal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?*

Solution. We prove that this is true by contradiction. Suppose that there is an $N \in \mathbb{N}$ such that $S(N) < 4N/5$ and $S(N+1) > 4(N+1)/5$. Such an N exists since $S(N)$ starts below 80% of N and ends above 80% of N without ever being exactly 80%. Suppose that she makes m of her first N free throws. $S(N) < S(N+1) \leq S(N)+1$ for otherwise $S(N+1) < 4N/5 \leq 4(N+1)/5$. In other words, she must make $m+1$ of her first $N+1$ free throws. This implies that $m/N < 4/5$ while $(m+1)/(N+1) > 4/5$. In other words $5m < 4N$ and $5(m+1) > 4(N+1)$. Rearranging gives that $5m < 4N < 5m+1$. But this is impossible since $4N$ is an integer between the consecutive integers $5m$ and $5m+1$.

Problem 2. *Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.*

Solution. Suppose otherwise. Each vertex is a vertex for five faces, all of which have different labels, and so the sum of the labels of the five faces incident to v is at least $0+1+2+3+4 = 10$. Adding this sum over all vertices gives at least $12 \times 10 = 120$. Note however that every face is incident to exactly three vertices, hence this sum is precisely three times the sum of all the faces $= 3 \times 39 = 117$. But $120 > 117$ which is a contradiction.

Problem 3. *Let $f(x) = 3x^2 + 1$. Prove that for any positive integer n, the product $f(1)f(2)\ldots f(n)$ has at most n prime divisors. (Bonus: Show that for $n \geq 4$ it has at most $n-1$ prime divisors).*

Solution. We prove a stronger statement. Namely, suppose that $n \geq 2$ and that $f(1)f(2)\ldots f(n-1)$ has k prime divisors. Then $f(1)f(2)\ldots f(n)$ has at most $k+1$ prime divisors. Indeed any prime divisor p of $f(1)f(2)\ldots f(n)$ which is not a divisor of $f(1)f(2)\ldots f(n-1)$ must satisfy $p \mid f(n)$ and $p \nmid f(i)$ for all $0 < i < n$. Since $p \mid f(n) = 3n^2+1$, we have $p \nmid n$. Moreover, $p \nmid f(n) - f(i) = 3n^2 - 3i^2 = 3(n-i)(n+i)$ for all $0 < i < n$. But this implies that $p \geq 2n$ since any integer $< 2n$ which is not n will be of the form $n \pm i$ for some $0 < i < n$. Suppose now that there are two distinct prime divisors p, q dividing $f(1)f(2)\ldots f(n)$ but not $f(1)f(2)\ldots f(n-1)$. Then $p, q \mid f(n)$ and $p, q \geq 2n$. But this implies that $pq \geq 4n^2 > 3n^2+1$ which contradicts the fact that $pq \mid 3n^2+1$.

To conclude the proof, note that $f(1) = 4$ has one prime divisor. We then proceed by induction using the induction step proved above to conclude that $f(1)f(2)\ldots f(n)$ has at most n prime divisors. We can also note that $f(1)f(2)f(3)f(4) = 2^4 \cdot 7^3 \cdot 13$ has 3 prime divisors, which proves the bonus problem.

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