UBC MATH CIRCLE 2024 PROBLEM SET 7 SOLUTIONS

Problem 1. *Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all P in the plane?*

Solution. Yes it does follow. Let P be any point in the plane and $ABCD$ be any square with center P. Let E,F,G,H be the midpoints of the segments AB,BC,CD,DA respectively. Note that we then have squares $ABCD$, $EFGH$, $AEPH$, $BFPE$, $CGPF$ and $DHPG$. Thus f must satisfy

$$\begin{aligned} f(A)+f(B)+f(C)+f(D)&=0\\ f(E)+f(F)+f(G)+f(H)&=0\\ f(A)+f(E)+f(P)+f(H)&=0\\ f(B)+f(F)+f(P)+f(E)&=0\\ f(C)+f(G)+f(P)+f(F)&=0\\ f(D)+f(H)+f(P)+f(G)&=0 \end{aligned}$$

Adding the last four equations, subtracting the first equation and twice the second equation gives $4f(P)=0$. Whence $f(P)=0$.

Problem 2. *Let m distinct positive integers $a_1,\dots,a_m$ be given. Prove that there exist fewer than 2^m positive integers $b_1,\dots,b_n$ such that all sums of distinct b_k's are distinct and all a_i for $1\le i\le m$ occur among them.*

Solution. Consider the set B of powers of two that appear in the binary expansion of a_i for some $i\in\{1,\dots,m\}$. For each subset $S\subset\{1,\dots,m\}$, define a subset $B_S\subset B$ as follows. A power of two, $b\in B$, is contained in B_S if it does not appear in the binary expansion of a_j for any $j\notin S$ and appears in the binary expansion of a_i for all $i\in S$. Note that for any $i\in\{1,\dots,m\}$, the set of powers of two appearing in the binary expansion of a_i is precisely $\bigcup_{i\in S}B_S$ where the union is taken over subsets $i\in S\subset\{1,\dots,m\}$. Indeed, no powers of two not contained in the binary expansion of a_i can lie in $\bigcup_{i\in S}B_S$, conversely, any power of two b in the binary expansion of a_i lies in B_T where $T=\{j\in\{1,\dots,m\}:$ b is in the binary expansion of $a_j\}$ and this contains i. Note moreover that for any distinct S,T, we have $B_S\cap B_T=\emptyset$. Indeed, if $b\in B_S$ and $i\in T\setminus S$ then $b\notin B_T$ since b is not a power of two in the binary expansion of a_i. Similarly, if $i\in S\setminus T$, then again $b\notin B_T$ since b is a power of two in the binary expansion of a_i, but no element of T is a power of two in the binary expansion of a_i. For each subset $S\subset\{1,\dots,m\}$ such that B_S is nonempty, let $b_S=\sum_{b\in B_S}b$. We have seen that, for any $i\in\{1,\dots,m\}$, the set of powers of two in the binary expansion of a_i is $\bigcup_{i\in S}B_S$ and that these are all disjoint, thus $a_i=\sum_{i\in S}b_S$. Thus the a_i occur among all sums of the $\{b_S : S\subset\{1,\dots,m\}$ with $b_S\neq\emptyset\}$. Moreover, since the powers of two in the binary expansions of the b_S are all disjoint, no two distinct sums of the b_S can be the same for otherwise there would be subsets $\{S_1,\dots,S_k\}$ and $\{T_1,\dots,T_l\}$ of

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$\{1, \ldots, m\}$ such that $\bigcup_{i=1}^{k} B_{S_i} = \bigcup_{j=1}^{l} B_{T_j}$ and disjointness implies that the two collections of subsets are the same. Finally, note that $B_{\emptyset} = \emptyset$, so that $\#\{b_S : S \subset \{1, \ldots, m\}$ with $b_S \neq \emptyset\} \leq \#\{S \subset \{1, \ldots, m\} : S \neq \emptyset\} = 2^m - 1$. Thus $\{b_S : S \subset \{1, \ldots, m\}$ with $b_S \neq \emptyset\}$ is the desired set.

Problem 3. *For any polynomial $P \in \mathbb{C}[x]$ and for each complex number a, denote by P_a the set of all $z_0 \in \mathbb{C}$ such that $P(z_0) = a$. Let $P, Q \in \mathbb{C}[x]$ such that $P_2 = Q_2$ and $P_5 = Q_5$. Prove that $P = Q$.*

Solution. Clearly this holds if P and hence Q are constant. Hence we may assume that P and hence also Q have degree at least 1. Let $\alpha_1, \ldots, \alpha_r$ be the elements of $P_2 = Q_2$ and $\beta_1, \ldots, \beta_s$ be the elements of $P_5 = Q_5$. Then we let $k_1, \ldots, k_r$ be the respective multiplicities of $\alpha_1, \ldots, \alpha_r$ as roots of $P(x) - 2$ and $m_1, \ldots, m_s$ be the respective multiplicities of $\beta_1, \ldots, \beta_s$ as roots of $P(x) - 5$. Then $k_1 - 1, \ldots, k_r - 1$ are the multiplicities of $\alpha_1, \ldots, \alpha_r$ as roots of $P'(x)$ and similarly $m_1 - 1, \ldots, m_s - 1$ are the multiplicities of $\beta_1, \ldots, \beta_s$ as roots of $P'(x) = 0$. Letting d be the degree of $P(x)$, we get

$$d - 1 \geq \sum_{i=1}^{r} (\alpha_i - 1) + \sum_{j=1}^{s} (\beta_j - 1)$$

On the other hand, we have $d = \sum_{i=1}^{r} \alpha_i$ and $d = \sum_{j=1}^{s} \beta_j$ since these are the multiplicities of the roots of $P(x) - 2$ respectively $P(x) - 5$ (recall that P has degree ≥ 1). So we obtain $d - 1 \geq 2d - r - s$ or equivalently $r + s \geq d + 1$. If D is the degree of $Q(x)$, then applying the same argument to Q gives that $r + s \geq D + 1$. In other words, the polynomial $P(x) - Q(x)$, whose degree is bounded above by $\max\{d, D\}$, has at least $r + s \geq \max\{d, D\} + 1$ roots $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ counted with multiplicity. But this implies that $P = Q$ identically.