UBC MATH CIRCLE 2024 PROBLEM SET 8 SOLUTIONS

Problem 1. *Find all solutions to the equation $E+V+F=G+2$ where E,V,F,G are positive integers and E,V,F all divide G.*

Solution. Since E,V,F all divide G, we can write $p=G/E$, $q=G/V$ and $r=G/F$ and study the equation

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{2}{G}+1$$

where $p,q,r \mid G$ are positive integers. We may additionally suppose without loss of generality that $p \leq q \leq r$. Then $p \leq 2$, for otherwise $1/p+1/q+1/r \leq 1 < 2/G+1$. If $p=1$, then this forces $q=r=G$ since both $q,r \leq G$. If $p=2$, then we have

$$\frac{1}{q}+\frac{1}{r}=\frac{2}{G}+\frac{1}{2}$$

and $q \leq 3$ for otherwise $1/q+1/r \leq 1/2 < 2/G+1/2$. If $q=2$, then $r=G/2$ (in particular G is even). Otherwise $q=3$ and $G=12r/(6-r)$. Since $G>0$ and $r \geq 3$, we can only have $r=3,4,5$, which give $G=12,24,60$ respectively. Thus the only solutions are

$$\begin{aligned}
1+1+n &= 2+n \qquad (n \in \mathbb{N})\\
n+n+2 &= 2+2n \qquad (n \in \mathbb{N})\\
4+4+6 &= 2+12\\
6+8+12 &= 2+24\\
12+20+30 &= 2+60
\end{aligned}$$

In the ADE classification, these are labelled by A_{n-1}, D_{n+2} and E_6, E_7, E_8 respectively.

Problem 2. *A* population *on a graph is an assignment of positive integers to each vertex of the graph. A* perfect population *has the property that the population of each vertex is exactly 1/2 of the sum of the neighbouring populations. Find all perfectly populated (finite) graphs.*

Solution. We will show that the following list give all finite connected graphs with nontrivial perfect populations.

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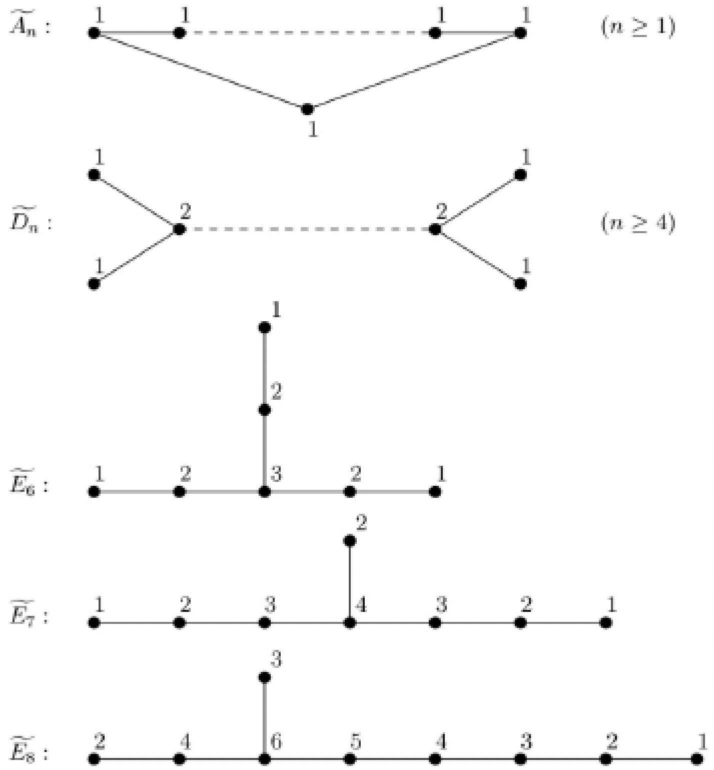


Indeed, these are obviously all perfect populations, hence we need only show the converse. Given a connected graph G with vertex set V and edge set E, define a matrix $A^G = \{A^G_{ij}\}_{i,j \in V}$ called the *adjacency matrix* as follows. $A^G_{ij} = 1$ if $i \neq j$ and $\{i, j\} \in E$, and 0 otherwise. Suppose that G has a perfect population. We may assume G is not trivial. Let $x = \{x_i\}_{i \in V}$ be the vector such that x_i is the population of vertex $i \in V$. By definition

$$\sum_{\{i,j\} \in E} x_i = 2x_j$$

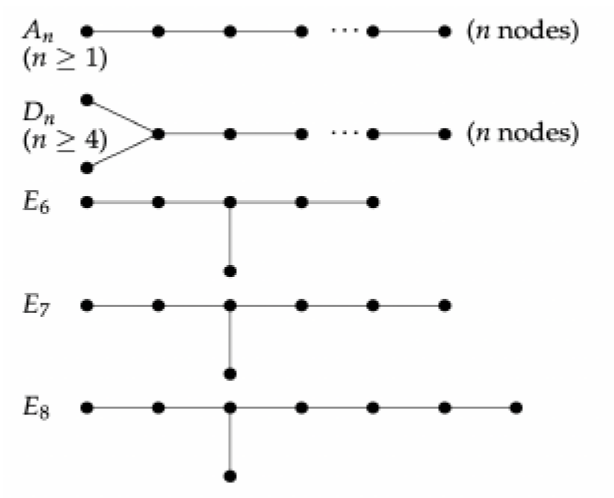
for all $j \in V$ and rewriting this with the matrix shows that $A^G x = 2x$. In other words x is a positive eigenvector of A^G of eigenvalue 2. We now state an important theorem which will be the basis of our proof.

Theorem 1 (Perron-Frobenius). *An $n \times n$ matrix A is* primitive *if there is a $k \in \mathbb{N}$ such that A^k has all positive entries. Let A be primitive matrix. Then*

(1) *There is a positive real number λ such that λ is a simple root of the characteristic polynomial of A (in particular λ is an eigenvalue of A) and for any other eigenvalue τ of A, we have $|\tau| < \lambda$.*
(2) *The eigenvector v of λ has only positive real entries and any other eigenvector w of A with positive real entries is a positive real multiple of v.*

We claim that A^G is primitive for any finite connected graph G which is not the trivial graph. Indeed, note that the (i, j) entry of $(A^G)^n$ is precisely the number of length n non-constant walks (i.e. the walk must change vertices at every step) from i to j in G. Since

G is connected and finite, there exists an $n \in \mathbb{N}$ such that for any pair of vertices $i, j \in V$, there is a walk from i to j. Hence the (i, j) entry is strictly positive for all $i, j \in V$ and A^G is primitive. In particular, since x is a positive eigenvector of eigenvalue 2, we conclude that the maximal eigenvalue of A^G is 2. Suppose conversely that A^G has maximal eigenvalue 2, then Perron-Frobenius applies again to show that A^G has a positive eigenvector x of eigenvalue 2 and hence x is a perfect population on G. Thus a connected nontrivial graph G has a perfect populations if and only if A^G has maximal eigenvalue 2. Suppose therefore that G is a connected graph such that A^G has maximal eigenvalue 2 and suppose additionally that $H \subset G$ is a subgraph where H belongs to the list above. Then since H has a perfect population, there is a vector x such that $A^H x = 2x$. Extending x to a vector on the vertex set of G by setting the remaining coordinates to 0, we find a vector y with nonnegative coefficients such that $A^G y = 2y$. If $H \neq G$, then y has at least one zero entry and it follows by Perron-Frobenius that y is not an eigenvector associated to the largest eigenvalue. Thus A^G has maximal eigenvalue > 2. Hence $H \subset G$ if and only if $H = G$. In particular, we may assume that $H \not\subset G$ for any H in the list above. Since G contains no $\widetilde{A}_n$, G is a tree. Since G doesn't contain the star graph $\widetilde{D}_4$ which has a vertex of degree 4, we conclude that every vertex of G has degree ≤ 3. Moreover, there is at most one vertex of degree 3 since G doesn't contain $\widetilde{D}_n$ for any n. Therefore G is either a path or has a unique vertex of degree 3 with three branches. Since G doesn't contain $\widetilde{E}_6, \widetilde{E}_7$ or $\widetilde{E}_8$, the only options are as follows.



But we can now check each of these graphs to see that for each of them A^G has a positive eigenvector of eigenvalue < 2. Hence by Perron-Frobenius, we conclude that each has maximal eigenvalue < 2. Hence if A^G does not have maximal eigenvalue 2, then G does not have a perfect population. We To show that these are the only perfect populations on these graphs, we apply Perron-Frobenius again to conclude that all eigenvectors associated to a maximal eigenvalue of A^G differ by multiplication by a constant. This concludes the proof.

Remark. One can also show this directly as follows. First show that a perfectly populated finite graph G which contains a cycle must equal the cycle, hence G is either a cycle or a tree. If G is cycle, it is straightforward to show that the only perfect population on G is the one above. Hence assume that G is a tree. Next show that G cannot be a path. Hence G has a vertex of degree ≥ 3. Show now that every vertex of G has degree ≤ 4 and that if G has a vertex of degree 4, then $G = \widetilde{D}_4$, with the associated perfect population. Hence we may assume that G has only vertices of degree ≤ 3. Now show that a sequence of vertices leaving a vertex of degree 3 must have populations which form a descending arithmetic progression. Hence if G has two vertices of degree 3, then the vertices between the two degree 3 vertices have constant population. Thus we get $\widetilde{D}_n$ in general. Hence we may assume that G has only a single vertex of degree 3. In other words G is a graph with one vertex of degree 3 and three strands off of it, each of which forms a descending arithmetic progression on population. A computation now shows that the only possibilities are $\widetilde{E}_6, \widetilde{E}_7$ and $\widetilde{E}_8$.