INSTRUCTIONS

SFU-UBC Math Contest

1. There are 4 sections:
   - Calculus
   - Linear Algebra
   - Discrete Mathematics
   - Number Theory
2. You (and your team) must pick one question from each of the 4 sections.
3. There are also several rules that you must follow:
   - The contest is open book and internet; however, you should not communicate with anyone outside of your registered team about the exam.
   - The solutions can use any material taught in some undergraduate courses, but cannot use material from pure graduate courses.
   - Once the contest ends, you will have 30 minutes to prepare your answers for submission. We are not receiving submissions after 4:00PM.
   - The contest organizers will be able to answer logistical questions on zoom during the contest time.
     - You can access the zoom by clicking here.
   - Please do not include your names in your submission. You should only include your team name in the email submission.
   - We will not be accepting regrade requests.
4. Submission Instructions:
   - You should submit their exam as a pdf document to one of these email addresses: ums.ubc@gmail.com or math-pres@sfu.ca.
   - Use your team name as the email title.

TWO CALCULUS RELATED QUESTIONS

V. JUNGIC

Problem 1. Draw a right triangle $\triangle A_0B_0C$ with the hypothenuse A_0C of length 1, and the angle $\angle CA_0B_0 = x$, $0 < x < \pi/2$.

For each $i = 0, 1, 2, 3\ldots$ recursively construct a sequence of points $A_i \in \overline{A_0C}$ and $B_i \in \overline{B_0C}$ so that $|\overline{A_{i-1}A_i}| = |\overline{A_{i-1}B_{i-1}}|$ and $\overline{A_iB_i} \perp \overline{B_0C}$.

Label the lengths of line segments: $a_i = |\overline{A_iC}|$, $b_i = |\overline{A_iB_i}|$, and the arc length $c_i = |\overset{\frown}{A_{i+1}B_i}|$. Finally, let d_i be the area of the sector $B_iA_iA_{i+1}$.

See Figure 1.

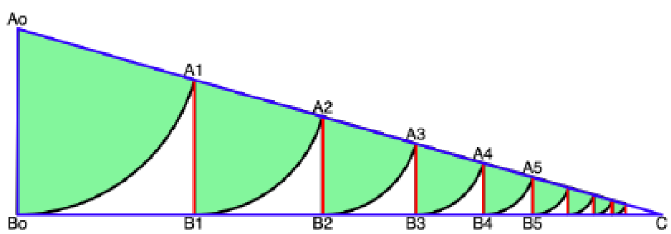

FIGURE 1. $\triangle A_0B_0C$ and points A_i and B_i, $i = 0, 1, 2, 3\ldots$

Find the following four quantities:

a) $a = \sum_{i=0}^{\infty} a_i$
b) $b = \sum_{i=0}^{\infty} b_i$
c) $c = \sum_{i=0}^{\infty} c_i$
d) $d = \sum_{i=0}^{\infty} d_i$

Problem 2. Consider the integral

$$I_k = \int_0^{\pi/2} x \cos^k x dx, \ \ k = 0, 1, 2, \ldots$$

a) Find a recursive formula for the sequence I_k, $k = 0, 1, 2, \ldots$
b) Use your result from a) to prove that

$$I_k = a_k\pi^2 + b_k\pi + c_k$$

for some rational numbers a_k, b_k, and c_k.

TWO LINEAR ALGEBRA RELATED QUESTIONS

B. Williams

Problem 1

There are 4 vessels (labeled 1, 2, 3, 4) containing 36 litres of water in total. Let x_i denote the quantity of water in vessel i, and suppose the water is distributed in such a way that $x_1 > x_2 > x_3 > x_4 \geq 0$. The following procedure is carried out at least once: the contents of the vessel with the most water is distributed evenly among the three vessels. It is observed afterwards that the amounts of water in the vessels is exactly as it was in the start: vessel 1 again has x_1 in it, vessel 2 again has x_2 etc.

Determine with proof the possible values for x_1.

Problem 2

Recall that the *trace* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the quantity $\mathrm{Tr}(A) = a + d$. It is also the case that $\mathrm{Tr}(A)$ is the sum of the eigenvalues of A, with multiplicity. Suppose that A and B are two 2×2-matrices with complex entries such that $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$ and

$$\mathrm{Tr}(A^2)\mathrm{Tr}(B^2) = \mathrm{Tr}(AB).$$

Prove that there exists a vector $\mathbf{v}$ that is an eigenvector of both A and B.

TWO DISCRETE MATH RELATED QUESTIONS

J. Solymosi

Problem 1

Let us consider the collection of all length three strings composed using r characters, where r is a positive, even integer. Now in each of the r^3 strings we erase one character. After that we are left with $m \leq r^2$ different strings of length two. The set of the m distinct strings is denoted by M. Give a lower bound on $|M| = m$ in terms of r and show that your bound is the best possible.

Problem 2

Let's place rooks on the field of the spatial $4 \times 4 \times 4$ chessboard. A rook attacks the 9 fields along the directions of the board edges plus the field it stands on. What is the minimal number of rooks necessary to attack all fields?

TWO NUMBER THEORY RELATED QUESTIONS

D. Ghioca

Problem 1

Show that there is no integer n larger than 1 with the property that n divides

$$625^{n-1} - 125^{n-1} + 25^{n-1} - 5^{n-1} + 1$$

Problem 2

Let $f \in \mathbb{Z}[x]$ be a polynomial of degree 2022 with integer coefficients. Show that there exist infinitely many positive integers n with the property that

$$\sqrt[5]{f(n)} \text{ is not an integer}$$