UBC-SFU discrete math questions

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1 First question

Let us consider the collection of all length three strings composed using r characters, where r is a positive, even integer. Now in each of the r^3 strings we erase one character. After that we are left with $m \leq r^2$ different strings of length two. The set of the m distinct strings is denoted by M. Give a lower bound on $|M| = m$ in terms of r and show that your bound is the best possible.

1.1 Solution

Let us denote the set of length three strings by S. The process of erasing one character defines a map $F : S \to M$, with $F(x, y, z) = (u, v)$ where (u, v) is $(x, y), (x, z)$ or (y, z). Let us partition the r^2 strings of length two into partition classes according to their first character. A sting, uv, is called empty if there is no string xyz such that $F(x, y, z) = (u, v)$. Let us choose a partition class with the most, say $\ell, (\ell \leq r)$, empty strings. These empty strings are $uv_1, uv_2, \ldots uv_\ell$. Since these strings are empty, $F(u, v_i, v_j) = (v_i, v_j)$ for all $1 \leq i, j \leq \ell$. In particular, these ℓ^2 strings, v_iv_j, are not empty. In each of the the $r - \ell$ partition classes where the first character is not v_i, $(1 \leq i \leq \ell)$, we have at least $r - \ell$ non-empty strings. Note that none of the k^2 nonempty strings is counted in the $r - \ell$ partition classes where the first character is not v_i. Therefore the number of non-empty strings, which is m, is at least

$$\ell^2 + (r - \ell)^2.$$

The minimum of the above formula is $r^2/2$ which is achieved when $r = 2\ell$.

To show that $r^2/2$ is possible, let us suppose that the characters are numbers form $\{0, 1, \ldots, r - 1\}$. From any string of length three let's erase one such that the sum of the remaining two characters is even. Then all strings uv where $u+v$ is odd will be empty.

2 Second question

Let's place rooks on the fields of the spatial $4 \times 4 \times 4$ chessboard. A rook attacks the 9 fields along the directions of the board edges plus the field it stands on. What is the minimal number of rooks necessary to attack all fields?

2.1 Solution

The answer is 8. Let's see first that 8 is enough. The levels (from bottom to top) where the rooks are placed are indicated by their number on the picture.

1			4
	3	2	
	2	3	
4			1

Every rook attacks 10 fields and there are 64 fields, so we need at least 7 rooks. Let us suppose that we have 7 rooks, attacking all fields. Now let's count how many fields are attacked by multiple rooks. If we can find at least 7 "over attacks" on seven or less fields then we are done, since we need at least 8 rooks ($70-7=63<64$). There are three times four distinct, 4×4 slices of the $4\times4\times4$ chessboard, determined by the planes orthogonal to the board edges. These are 4×4 planar chessboards. Any pair of rooks on the same slice determines at least two over attacks. If one of the 4×4 slices contains 4 or more rooks, then they determine more than 7 over attacks on this board already. If it contains three rooks, then they determine at least 6 over attacks. There is another slice parallel to this one containing at least two rooks, which determine at least two over attacks, so we are done. Thus we can suppose that no slice contains more than two rooks. Any pair of rooks on the same slice determines at least two over attacks. If they attack each other then it causes 4 over attacks, but in this case the pair is in two slices. In all three directions we have one slice with one rook and three with two rooks. So, the number of over attacks on boards orthogonal to one direction is $(2+2+2)=6$, if no two rooks attack each other and at least 8 if there is an attacking pair on one of the slices (which we can exclude now). The same holds to the slices on the other two directions. Let's choose one of them. Any over attack here, in these slices, is different from the previously counted, so they add at least $1+1+1=3$ over attacks to the previous 6, and we are done.